

Point-shifts of Point Processes on Topological Groups

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Abstract

This paper focuses on covariant point-shifts of point processes on topological groups, which map points of a point process to other points of the point process in a translation invariant way. Foliations and connected components generated by point-shifts are studied, and the cardinality classification of connected components, previously known on Euclidean space, is generalized to unimodular groups. An explicit counterexample is also given on a non-unimodular group. Isomodularity of a point-shift is defined and identified as a key component in generalizations of Mecke's invariance theorem in the unimodular and non-unimodular cases. Isomodularity is also the deciding factor of when the reciprocal and reverse of a point-map corresponding to a bijective point-shift are equal in distribution. Finally, sufficient conditions for separating points of a point process are given.

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1. Introduction

1.1. Outline

A review of what is necessary from the Palm theory of stationary random measures on locally compact second-countable Hausdorff groups is given. G. Last studied the Palm theory of locally compact second-countable Hausdorff groups in [7] and [8], though only the finite intensity case is considered here. Generally speaking, the mass transport theorem is used to study point-shifts of point processes, which were also studied on \mathbb{R}^d by F. Baccelli and M.-O. Haji-Mirsadeghi in [2] and [1].

The novelty of this document is essentially summarized in the following four loosely stated results, though many results leading up to them are interesting in their own right.

Theorem 1.1.1. *On unimodular groups, the cardinality classification of connected components of the graph generated by a point-shift holds. The graph is locally finite and flow-adapted. A component C is of type \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{F} , or \mathcal{I}/\mathcal{I} with \mathcal{I} (infinite) and \mathcal{F} (finite) representing the cardinality of the component/all foils therein. A component is acyclic iff it is infinite iff there is a flow-adapted linear order isomorphic to the order of \mathbb{N} or \mathbb{Z} on the foils. There are primeval elements, i.e. elements with preimages of all orders, iff there is a finite foil in C , and when they exist the primeval elements form either a unique cycle or unique bi-infinite path in C .*

Example 1.1.2. There exists an explicit non-unimodular group and point-shift \mathfrak{H} for which the previous cardinality classification fails in many respects. The graph generated by \mathfrak{H} is not locally finite. Every component consists of a single foil and is infinite (class \mathcal{I}/\mathcal{I}), but the components are cyclic and each component contains a unique element with preimages of all orders, which is a fixed point of \mathfrak{H} .

The classification theorem is precisely stated as Theorem 2.1.11 and Section 2.1 builds up to it. The counterexample showing the classification does not hold for non-unimodular groups follows in Section 2.2.

Theorem 1.1.3. *On unimodular groups, Mecke's invariance theorem holds. Point-shifts are bijective iff they preserve Palm probabilities. On possibly non-unimodular groups, the class of bijective point-shifts that preserve Palm probabilities is identified as the bijective isomodular point-shifts, the point-shifts that preserve the modular function of the group. The bijective isomodular point-shifts are also exactly the point-shifts for which the reciprocal and the reverse of the corresponding point-map are equal in distribution.*

Section 3.1 and Section 3.2 cover the previous theorem. Mecke's invariance theorem in the unimodular case is Corollary 3.1.1, and the identification of isomodular point-shifts as being the ones that preserve Palm probabilities (which may be considered a generalization of Mecke's invariance theorem for the non-unimodular case) is Theorem 3.1.6. The study of the reciprocal and reverse of a point-map and how isomodularity plays a role culminates in Theorem 3.2.4.

Theorem 1.1.4. *Given a measurable property $F(x, y_1, \dots, y_n)$ of a configuration of $n + 1$ points, if for every choice of distinct y_1, \dots, y_n , none of which are the identity, the set of x for which $F(x, y_1, \dots, y_n) \in M$ has null Haar measure, then no stationary point process has $n + 1$ distinct points X, Y_1, \dots, Y_n satisfying $F(X, Y_1, \dots, Y_n) \in M$. This generalizes fact on \mathbb{R}^d that a stationary point process has not two points X, Y that are equidistant from the origin.*

Section 3.3 studies when functions separate points of a point process, and the precise version of the previous theorem is Theorem 3.3.2.

1.2. Preliminaries

Throughout this document, let \mathbb{X} be a fixed locally compact second-countable Hausdorff topological group with identity element $e \in \mathbb{X}$. It will always be assumed that \mathbb{X} and \mathbb{R} are equipped with their respective Borel σ -algebras $\mathcal{B}(\mathbb{X})$ and $\mathcal{B}(\mathbb{R})$.

Let λ denote a fixed left-invariant **Haar measure** on \mathbb{X} , i.e. λ is a nontrivial locally finite Borel measure satisfying $\lambda(xB) = \lambda(B)$ for all $x \in \mathbb{X}, B \in \mathcal{B}(\mathbb{X})$. Also let $\Delta : \mathbb{X} \rightarrow (0, \infty)$ denote the **modular function** of \mathbb{X} . That is, $\lambda(Bx) = \Delta(x)\lambda(B)$ for all $x \in \mathbb{X}, B \in \mathcal{B}(\mathbb{X})$. Recall that Δ is a continuous homomorphism, i.e. $\Delta(xy) = \Delta(x)\Delta(y)$, and from this follows $\Delta(x^{-1}) = \Delta(x)^{-1}$ for all $x, y \in \mathbb{X}$.

Let \mathbf{M} be the space of all **Radon measures** on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ equipped with the Borel σ -algebra \mathcal{M} induced by the topology of vague convergence. The shift operators $\{\theta_x\}_{x \in \mathbb{X}}$ act on elements of \mathbf{M} by $\theta_x \mu(B) := \mu(x^{-1}B)$. The zero measure on \mathbb{X} is denoted $\mathbf{0}$. The **support** of μ is denoted $\bar{\mu}$. Also denote by \mathbf{N} the subset of **counting measures**. A counting measure is **simple** if each atom has mass 1, and a simple counting measure will be identified with its support, which is a discrete subset of \mathbb{X} .

A **random measure** is a pair (\mathbf{m}, \mathbf{P}) where $\mathbf{m} : (\Omega, \mathcal{A}) \rightarrow (\mathbf{M}, \mathcal{M})$ is a measurable mapping and $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space. Often $\mathbf{m}(\omega, B)$ is written instead of $\mathbf{m}(\omega)(B)$ for convenience, but this is also in accordance with the viewpoint of \mathbf{m} being a transition kernel. A **(simple) point process** is a random measure that is almost surely a (simple) counting measure.

The underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is always assumed to be part of a **stationary framework** $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbf{P})$ and any random measure \mathbf{m} will always be assumed to be **compatible** with the flow $\{\theta_x\}_{x \in \mathbb{X}}$. That is, suppose \mathbf{P} is invariant with respect to a measurable flow $\{\theta_x\}_{x \in \mathbb{X}}$ that acts on the underlying measurable space (Ω, \mathcal{A}) :

- (i) $\theta_x : \Omega \rightarrow \Omega$ for all $x \in \mathbb{X}$,
- (ii) θ_e is the identity on Ω ,
- (iii) $\theta_{xy} = \theta_x \theta_y$ for all $x, y \in \mathbb{X}$, in particular $\theta_x^{-1} = \theta_{x^{-1}}$,
- (iv) $(\omega, x) \mapsto \theta_x \omega$ is $\mathcal{B}(\mathbb{X}) \otimes \mathcal{A}$ -measurable, and
- (v) $\mathbf{P} \circ \theta_x^{-1} = \mathbf{P}$ for all $x \in \mathbb{X}$,

and it is assumed that the random measure \mathbf{m} on \mathbb{X} satisfies

$$\mathbf{m}(\theta_x \omega, B) = \mathbf{m}(\omega, x^{-1}B), \quad x \in \mathbb{X}, \omega \in \Omega, B \in \mathcal{B}(\mathbb{X}).$$

Abusing notation, this is written as

$$\mathbf{m}(\theta_x \omega, B) = \theta_x \mathbf{m}(\omega, B).$$

This is an abuse of notation because on the left side the notation θ_x means the measurable flow on Ω , and on the right side θ_x means the shift operator on \mathbf{M} . Under these assumptions any such \mathbf{m} is stationary in the usual sense that \mathbf{m} and $\theta_x \mathbf{m}$ have the same distribution for all $x \in \mathbb{X}$.

Example 1.2.1. If \mathbf{m} is a random measure on \mathbb{X} on the canonical space with $(\Omega, \mathcal{A}, \mathbf{P}) := (\mathbf{M}, \mathcal{M}, \mathcal{P})$ where $\mathbf{m}(\mu) := \mu$ is the identity map on \mathbf{M} and \mathcal{P} is the distribution of \mathbf{m} , then $\theta_x \mu$ defined to be the shift operator $\theta_x \mu(B) := \mu(x^{-1}B)$ makes $(\mathbf{M}, \mathcal{M}, \{\theta_x\}_{x \in \mathbb{X}}, \mathcal{P})$ a stationary framework when the distribution of \mathbf{m} is shift-invariant.

It is also assumed that all random measures \mathbf{m} that are introduced have finite and nonzero intensity, usually denoted γ . It holds that $\gamma = \frac{\mathbf{E}[\mathbf{m}(B)]}{\lambda(B)}$ for any B with finite and nonzero Haar measure.

The **Palm probability** of \mathbf{m} is denoted $\mathbf{P}^{\mathbf{m}}$, i.e. for all $A \in \mathcal{A}$,

$$\mathbf{P}^{\mathbf{m}}(A) := \frac{1}{\gamma} \mathbf{E} \int_{\mathbb{X}} 1_{\theta_x^{-1} \in A} w(x) \mathbf{m}(dx)$$

for any (and every) measurable $w : \mathbb{X} \rightarrow \mathbb{R}_+$ with $\int_{\mathbb{X}} w d\lambda = 1$. It is typical to take $w(x) := \frac{1}{\lambda(B)} 1_{x \in B}$ for some B with $0 < \lambda(B) < \infty$. Expectation with respect to $\mathbf{P}^{\mathbf{m}}$ is denoted $\mathbf{E}^{\mathbf{m}}$.

The Palm probability $\mathbf{P}^{\mathbf{m}}$ gives the view of the world from a typical point's perspective. Intuitively it is the reference measure conditioned on the event that $e \in \bar{\mathbf{m}}$. Indeed, $\mathbf{P}^{\mathbf{m}}(e \in \bar{\mathbf{m}}) = 1$ and the interpretation as a conditional expectation is exactly correct when \mathbb{X} is discrete.

The connection between \mathbf{P} and $\mathbf{P}^{\mathbf{m}}$ is given by the refined Campbell theorem, abbreviated to C-L-M-M for Campbell, Little, Mecke, and Matthes.

Theorem 1.2.2 (C-L-M-M). [7] *For all $f : \Omega \times \mathbb{X} \rightarrow \mathbb{R}_+$ measurable,*

$$\mathbf{E} \int_{\mathbb{X}} f(\theta_x^{-1}, x) \mathbf{m}(dx) = \gamma \mathbf{E}^{\mathbf{m}} \int_{\mathbb{X}} f(\theta_e, x) \lambda(dx).$$

It is also possible to convert between \mathbf{P} -a.s. and $\mathbf{P}^{\mathbf{m}}$ -a.s. events in the following manner.

Theorem 1.2.3. *Let $A \in \mathcal{A}$. Then the following are equivalent:*

- (a) $\mathbf{P}^{\mathbf{m}}(A) = 1$,
- (b) $\mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0) = 1$,
- (c) $\mathbf{P}^{\mathbf{m}}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0) = 1$.

The proof of this fact is given in the appendix. When \mathbf{m} is a point process, Theorem 1.2.3 can be used to translate between definitions under $\mathbf{P}^{\mathbf{m}}$ and definitions under \mathbf{P} . The unfamiliar reader should see Example 4.2.6 for the details of how to do this.

The primary tool that allows the study of point-shifts in this paper is the mass transport theorem.

Theorem 1.2.4 (Mass Transport Theorem). *[7] Suppose \mathbf{m}, \mathbf{m}' are compatible random measures on \mathbb{X} with respective intensities $\gamma, \gamma' \in (0, \infty)$. Then for all **diagonally invariant** τ , i.e. measurable $\tau : \Omega \times \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ invariant in the sense that*

$$\tau(\theta_z \omega, zx, zy) = \tau(\omega, x, y) =: \tau(x, y), \quad \omega \in \Omega, \quad x, y, z \in \mathbb{X}, \quad (1)$$

it holds that

$$\gamma \mathbf{E}^{\mathbf{m}} \int_{\mathbb{X}} \tau(e, y) \mathbf{m}'(dy) = \gamma' \mathbf{E}^{\mathbf{m}'} \int_{\mathbb{X}} \tau(x, e) \Delta(x^{-1}) \mathbf{m}(dx). \quad (2)$$

Interpret $\tau(\omega, x, y)$ as the amount of mass sent from x to y on the outcome ω . Under $\mathbf{E}^{\mathbf{m}}$, e is a point of \mathbf{m} . Thus the left side of (2) is an average of mass sent out of $e \in \mathbf{m}$ to all points \mathbf{m}' . Under $\mathbf{E}^{\mathbf{m}'}$, e is a point of \mathbf{m}' . Thus the right side of (2) is a weighted average of mass received by $e \in \mathbf{m}'$ from all points of \mathbf{m} . If $\Delta(x) = 1$ for all $x \in \mathbb{X}$, i.e. if \mathbb{X} is unimodular, then the mass transport formula is the one expected from the case of translations on \mathbb{R}^d , which says that after weighting by the ratio of intensities γ/γ' , the mass a typical point of \mathbf{m}' receives is equal to the mass that a typical point of \mathbf{m} sends, on average.

When dealing with point-shifts and point-maps, it will always be assumed that \mathbf{m} is a simple point process. A **point-shift** on \mathbf{m} is a covariant measurable map $\mathfrak{H} : \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ on the support of \mathbf{m} , i.e. for all $x, y \in \mathbb{X}, \omega \in \Omega$, \mathfrak{H} satisfies $\mathfrak{H}(\theta_y \omega, yx) = y\mathfrak{H}(\omega, x)$ and \mathbf{P} -a.e. $\omega \in \Omega$ is such that $\mathfrak{H}(\omega, X) \in \mathbf{m}(\omega)$ for all $X \in \mathbf{m}(\omega)$. If unspecified, $\mathfrak{H}(\omega, x) := x$ for $x \notin \mathbf{m}(\omega)$. Dependence on ω is usually dropped and $\mathfrak{H}(X)$ is written instead of $\mathfrak{H}(\omega, X)$. Say that \mathfrak{H} has a functional property, e.g. bijectivity, injectivity, surjectivity, if for \mathbf{P} -a.e. $\omega \in \Omega$, $\mathfrak{H}(\omega, \cdot)$ has the property on the support of $\mathbf{m}(\omega)$.

A **point-map** on \mathbf{m} is a measurable map $\mathfrak{h} : \Omega \rightarrow \mathbb{X}$ such that $\mathfrak{h}(\omega) \in \mathbf{m}(\omega)$ for $\mathbf{P}^{\mathbf{m}}$ -a.e. $\omega \in \Omega$.

There is a natural correspondence between point-shifts and point-maps. Namely, if \mathfrak{H} is a point-shift, then $\mathfrak{h}(\omega) := \mathfrak{H}(\omega, e)$ is a point-map, and if \mathfrak{h} is

a point-map, then $\mathfrak{H}(\omega, X) := X\mathfrak{h}(\theta_X^{-1}\omega)$ is a point-shift, and these operations are inverses.

To any point-shift \mathfrak{H} with corresponding point-map \mathfrak{h} on \mathfrak{m} define the functions (omitting ω dependence):

- **Edge indicator:** $\tau^{\mathfrak{H}}(x, y) := 1_{x, y \in \mathfrak{m}, \mathfrak{H}(x)=y}$ for all $x, y \in \mathbb{X}$.
- **Out-neighbors** and **in-neighbors** of e under $\mathbf{P}^{\mathfrak{m}}$:

$$\begin{aligned} h^+ &:= \{Y \in \mathfrak{m} : \tau^{\mathfrak{H}}(e, Y) = 1\} = \{\mathfrak{h}\}, \\ h^- &:= \{X \in \mathfrak{m} : \tau^{\mathfrak{H}}(X, e) = 1\} = \{Y \in \mathfrak{m} : \mathfrak{h}(\theta_Y^{-1}\omega) = Y^{-1}\}. \end{aligned}$$

- **Out-neighbors** and **in-neighbors** under \mathbf{P} or $\mathbf{P}^{\mathfrak{m}}$:

$$\begin{aligned} H^+(X) &:= Xh^+(\theta_X^{-1}) = \{Y \in \mathfrak{m} : \tau^{\mathfrak{H}}(X, Y) = 1\} = \{\mathfrak{H}(X)\}, \\ H^-(X) &:= Xh^-(\theta_X^{-1}) = \{Y \in \mathfrak{m} : \tau^{\mathfrak{H}}(Y, X) = 1\} = \{Y \in \mathfrak{m} : \mathfrak{H}(Y) = X\} \end{aligned}$$

for all $X \in \mathfrak{m}$.

- **Preimage** of e under $\mathbf{P}^{\mathfrak{m}}$: if $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\text{card}(h^-) = 1$, then \mathfrak{h}^- is defined to be the unique element in h^- . By the upcoming Proposition 1.2.5 this is equivalent to \mathfrak{H} being bijective.
- **Reverse point-shift:** if \mathbf{P} -a.s. $\text{card}(H^-(X)) = 1$ for all $X \in \mathfrak{m}$ (equivalently $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\text{card}(h^-) = 1$), then $\mathfrak{H}^-(X)$ is defined to be the unique element in $H^-(X)$. By the upcoming Proposition 1.2.5, \mathfrak{H}^- is defined iff \mathfrak{H} is bijective. In this case \mathbf{P} -a.s. $\mathfrak{H}(\mathfrak{H}^-(X)) = \mathfrak{H}^-(\mathfrak{H}(X)) = X$ for all $X \in \mathfrak{m}$, by definition. That is, \mathfrak{H} and \mathfrak{H}^- are inverses.

With these definitions, $\tau^{\mathfrak{H}}$ is diagonally invariant, H^+ , H^- , \mathfrak{H}^- are covariant with the flow, and the definitions of h^+ , h^- , \mathfrak{h}^- under $\mathbf{P}^{\mathfrak{m}}$ are equivalent to the definitions of H^+ , H^- , \mathfrak{H}^- under \mathbf{P} or $\mathbf{P}^{\mathfrak{m}}$ as in Example 4.2.6.

With the mass transport theorem and Theorem 1.2.3, the following can be obtained by chasing definitions:

Proposition 1.2.5. *The following hold:*

- \mathbf{P} -a.s. every $X \in \mathfrak{m}$ is the image under \mathfrak{H} of at least (resp. at most) k distinct points of \mathfrak{m} iff $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\text{card}(h^-) \geq k$ (resp. $\leq k$),
- \mathbf{P} -a.s. every $X \in \mathfrak{m}$ is the image under \mathfrak{H} of finitely (resp. infinitely) many distinct points of \mathfrak{m} iff $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\text{card}(h^-) < \infty$ (resp. $= \infty$),
- \mathbf{P} -a.s. \mathfrak{H} is bijective (resp. surjective, injective) iff $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\text{card}(h^-) = 1$ (resp. $\geq 1, \leq 1$). In particular \mathfrak{h}^- and \mathfrak{H}^- are well defined iff \mathfrak{H} is bijective,
- For all $f : \Omega \rightarrow \mathbb{R}_+$ measurable,

$$\mathbf{E}^{\mathfrak{m}}[f(\theta_{\mathfrak{h}}^{-1})] = \mathbf{E}^{\mathfrak{m}}[f \text{card}(h^-)], \quad (3)$$

- (e) **P**-a.s. every $X \in \mathfrak{m}$ is the image under \mathfrak{H} of at least (resp. at most) k points of \mathfrak{m} iff for all $f : \Omega \rightarrow \mathbb{R}_+$ measurable

$$\mathbf{E}^{\mathfrak{m}}[f(\theta_{\mathfrak{h}}^{-1})\Delta(\mathfrak{h}^{-1})] \geq k\mathbf{E}^{\mathfrak{m}}[f] \quad (\text{resp. } \leq k\mathbf{E}^{\mathfrak{m}}[f]),$$

- (f) (Test for Bijectivity)¹ \mathfrak{H} is bijective iff for all $f : \Omega \rightarrow \mathbb{R}_+$ measurable

$$\mathbf{E}^{\mathfrak{m}}[f(\theta_{\mathfrak{h}}^{-1})\Delta(\mathfrak{h}^{-1})] = \mathbf{E}^{\mathfrak{m}}[f], \quad (4)$$

- (g) If \mathfrak{H} is bijective, also

$$\mathbf{E}^{\mathfrak{m}} \left[f(\theta_{\mathfrak{h}}^{-1}) \right] = \mathbf{E}^{\mathfrak{m}} \left[\frac{f}{\Delta(\mathfrak{h}^{-})} \right], \quad (5)$$

- (h) If **P**-a.s. every $X \in \mathfrak{m}$ is the image under \mathfrak{H} of at least (resp. at most) k points of \mathfrak{m} , then $\mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})] \geq k$ (resp. $\leq k$),

- (i) If $\mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})] < \infty$, every $X \in \mathfrak{m}$ is the image of only finitely many $Y \in \mathfrak{m}$ under \mathfrak{H} ,

- (j) If $\mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})] = 1$, then \mathfrak{H} is injective iff it is surjective. In particular, this is automatic if \mathbb{X} is unimodular.

Proof. The proofs are sketched.

(a),(b),(c): Direct application of Theorem 1.2.3.

(d): Apply the mass transport theorem with the diagonally invariant function

$$\tau(\omega, x, y) := f(\theta_y^{-1}\omega)1_{x, y \in \mathfrak{m}(\omega), y = \mathfrak{H}(x)}\Delta(y^{-1}x).$$

(e): Apply (a) and (d).

(f): Apply (e) with $k := 1$.

(g): Replace f with $\frac{f}{\Delta(\mathfrak{h}^{-})}$ in (d) and use the fact that **P**^m-a.s.

$$\mathfrak{h}^{-}(\theta_{\mathfrak{h}}^{-1}) = \mathfrak{h}^{-1}(\mathfrak{h}\mathfrak{h}^{-}(\theta_{\mathfrak{h}}^{-1})) = \mathfrak{h}^{-1}\mathfrak{H}^{-}(\mathfrak{h}) = \mathfrak{h}^{-1}\mathfrak{H}^{-}(\mathfrak{H}(e)) = \mathfrak{h}^{-1}.$$

(h),(i): Take $f := 1$ in (d) and apply (a) or (b).

(j): Take $f := 1$ in (d), use (c) and the fact that a random variable bounded above (or below) by 1 with expectation 1 must be constant 1 a.s.

□

¹G. Last also proves this and similar results, e.g. Corollary 10.1 in [7].

1.3. Examples of Point-shifts

A few elementary examples of point-shifts on a compatible simple point process \mathfrak{m} are given. Distances in \mathbb{X} will be measured with respect to any left-invariant metric d inducing the topology of \mathbb{X} .

1. (Closest Neighbor Shift). For $X \in \mathfrak{m}$, let $\mathfrak{H}(X)$ be the closest $Y \in \mathfrak{m}$ with $Y \neq X$ (if one is uniquely determined), otherwise let $\mathfrak{H}(X) := X$.
2. (Mutual Closest Neighbor Shift). If $X, Y \in \mathfrak{m}$ are such that X is the closest neighbor to Y and Y is the closest neighbor to X , set $\mathfrak{H}(X) := Y$ and $\mathfrak{H}(Y) := X$. For $X \in \mathfrak{m}$ not mutually closest neighbors with another point, set $\mathfrak{H}(X) := X$.
3. (Closest Neighbor from Next Generation). Suppose \mathfrak{m} is partitioned into sub-processes $\mathfrak{m} = \sum_{i \in \mathbb{Z}} \mathfrak{m}_i$, then for $X \in \mathfrak{m}_i$, let $\mathfrak{H}(X)$ be the closest point $Y \in \mathfrak{m}_{i+1}$ (if one is uniquely determined), otherwise $\mathfrak{H}(X) := X$.
4. (Modularity Boost). Suppose \mathbb{X} is not unimodular. Fix a relatively compact set $B \in \mathcal{B}(\mathbb{X})$ containing the identity. For each $X \in \mathfrak{m}$, look in the set XB and set $\mathfrak{H}(X)$ to be the $Y \in \mathfrak{m} \cap XB$ such that $\Delta(Y)$ is maximum (or $\mathfrak{H}(X) := X$ if such a Y is not uniquely determined). Existence of a unique maximum can be guaranteed by assuming $\lambda(\Delta = 1) = 0$ if \mathfrak{m} is Poisson (see Corollary 3.3.6 and use the Slivnyak-Mecke theorem).
5. (Snap-to-grid Shift on Circle Group). Let $\mathbb{X} := \{z \in \mathbb{C} : |z| = 1\}$ be the circle group, which is identified with $[0, 1) \subseteq \mathbb{R}$ via $x \mapsto e^{2\pi i x}$. Since \mathbb{X} is Abelian it is unimodular, and its Haar measure is normalized arc length along the circle, or Lebesgue measure on $[0, 1)$. Let \mathfrak{m} be a homogeneous Poisson point process on \mathbb{X} with intensity $\gamma \in (0, \infty)$. Necessarily \mathfrak{m} is stationary and simple. Imagine picking up $X \in \mathfrak{m}$, rotating by an angle θ , and dropping X at the nearest point of \mathfrak{m} . More precisely, with $z := e^{2\pi i \theta} \in \mathbb{X}$,

$$\mathfrak{H}(X) := \arg \min_{Y \in \mathfrak{m}} |Y - zX|, \quad X \in \mathfrak{m}, \mathbf{P}\text{-a.s.}$$

Such a nearest point will be uniquely determined in this case (see again Corollary 3.3.6 and use the Slivnyak-Mecke Theorem).

6. (Strip Point-shift on the $ax + b$ Group). Recall the standard first example of a non-unimodular group: the $ax + b$ group. Let

$$\mathbb{X} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$$

with matrix multiplication and the topology inherited from \mathbb{R}^4 . \mathbb{X} is identified with the right half-plane in \mathbb{R}^2 by identifying (a, b) with $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

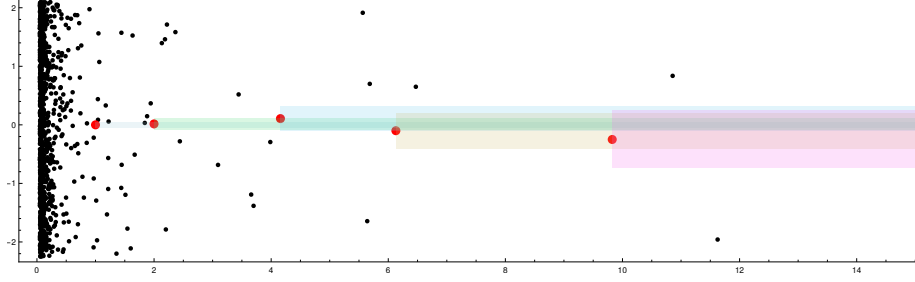


Figure 1: Iterates of e under the strip point-shift on the $ax + b$ group.

In this notation $(a, b)(c, d) = (ac, ad + b)$ and $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$. Then, cf. [5] Example 15.17 (g), \mathbb{X} has a left-invariant Haar measure

$$\lambda(B) = \iint_B \frac{1}{a^2} da db$$

and modular function

$$\Delta(a, b) = \frac{1}{a}.$$

Let \mathfrak{m} be a homogeneous Poisson point process on \mathbb{X} with intensity $\gamma \in (0, \infty)$. Necessarily \mathfrak{m} is stationary and simple. For all $(a, b) \in \mathbb{X}$ define the strip

$$S(a, b) := [a, \infty) \times [b - \delta a, b + \delta a]$$

for some fixed $\delta > 0$. Note that the definition is chosen so $(a, b)S(1, 0) = S(a, b)$, where here $(1, 0) = e \in \mathbb{X}$. Moreover, for any $(a, b) \in \mathbb{X}$,

$$\lambda(S(a, b)) = \int_{b-\delta a}^{b+\delta a} \int_a^\infty \frac{1}{x^2} dx dy = \frac{1}{a} \cdot ((b + \delta a) - (b - \delta a)) = 2\delta$$

so in particular $\mathfrak{m}(S(a, b)) < \infty$ a.s. By Theorem 4.2.7 $\mathfrak{m}^!$ is Poisson under $\mathbf{P}^{\mathfrak{m}}$ with $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(B)] = \gamma\lambda(B)$. Hence $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(S(1, 0))] = 2\delta\gamma$ and therefore $\mathfrak{m}(S(1, 0)) < \infty$, $\mathbf{P}^{\mathfrak{m}}$ -a.s. Equivalently, \mathbf{P} -a.s. $\mathfrak{m}(S(X)) < \infty$ for all $X \in \mathfrak{m}$ by Theorem 1.2.3. This leads to the **strip point-shift** \mathfrak{H} where $\mathfrak{H}(X)$ is defined to be the right-most point of \mathfrak{m} in $S(X)$ for each $X \in \mathfrak{m}$. Note that there is no need to resolve ties for right-most position because $\lambda(\{1\} \times \mathbb{R}) = 0$, so that $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(\{1\} \times \mathbb{R})] = 0$ and hence all $X \in \mathfrak{m}$ will have distinct first coordinates by Proposition 3.3.5.

2. Point-shift Foliations

2.1. The Cardinality Classification of Components

In this section the cardinality classification components of point-shifts in [2] is extended to the general stationary framework for unimodular \mathbb{X} . The classification theorem is Theorem 2.1.11, and the fundamental result used in its

proof, which says it is impossible to pick out finite subsets of infinite sets in a flow-adapted manner, is Theorem 2.1.6.

Throughout this section, \mathbb{X} is assumed to be unimodular. Fix a stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbf{P})$, a compatible simple point process \mathbf{m} on \mathbb{X} with intensity $\gamma \in (0, \infty)$, and a point-map \mathfrak{h} on \mathbf{m} with corresponding point-shift \mathfrak{H} . The wording of proofs is substantially cut down by thinking of $\mathfrak{H}(X)$ as the **father** of X . For example, the **children** of X are the $Y \in \mathbf{m}$ such that $\mathfrak{H}(Y) = X$. Next appear the necessary ingredients needed for the classification theorem.

Definition 2.1.1. The iterates \mathfrak{H}^n are defined by repeatedly applying the point-shift \mathfrak{H} . That is, $\mathfrak{H}^0(X) := X$ and $\mathfrak{H}^{n+1}(X) := \mathfrak{H}(\mathfrak{H}^n(X))$ for all $X \in \mathbf{m}$. Elements $Y \in \mathbf{m}$ that are in the image $\mathfrak{H}^n(\mathbf{m})$ for all $n \in \mathbb{N}$ are called **primeval**, and $\mathfrak{H}^\infty(\mathbf{m})$ will denote the set of all primeval elements of \mathbf{m} . Here $\mathfrak{H}^n(\mathbf{m})$ is considered as a set, i.e. multiplicities are ignored, for all $n \leq \infty$. The set $\mathfrak{H}^n(\mathbf{m})$ is flow-adapted for any $n \leq \infty$ because \mathfrak{H} is flow-adapted.

Definition 2.1.2. The random graph $G^\mathfrak{H}$ has vertices \mathbf{m} and directed edges from each $X \in \mathbf{m}$ to $\mathfrak{H}(X)$. The set of its undirected connected components is denoted by $\mathcal{C}^\mathfrak{H}$ and the component of $X \in \mathbf{m}$ is denoted $C^\mathfrak{H}(X)$. Then $X, Y \in \mathbf{m}$ are in the same component iff there are $n, m \in \mathbb{N}$ such that $\mathfrak{H}^m(X) = \mathfrak{H}^n(Y)$. That is, $C^\mathfrak{H}(X)$ is the set of all **relatives** of X . The graph $G^\mathfrak{H}$ is flow-adapted, and hence so is $\mathcal{C}^\mathfrak{H}$.

Definition 2.1.3. The **foliation** $\mathcal{L}^\mathfrak{H}$ is defined to be the set of **foils** $L^\mathfrak{H}(X)$ of \mathfrak{H} for $X \in \mathbf{m}$, which are equivalence classes under the equivalence relation where $X, Y \in \mathbf{m}$ are equivalent iff there is $n \in \mathbb{N}$ such that $\mathfrak{H}^n(X) = \mathfrak{H}^n(Y)$. That is, $L^\mathfrak{H}(X)$ is the relatives of X from the same **generation** as X . The foliation $\mathcal{L}^\mathfrak{H}$ is flow-adapted, and $\mathcal{L}^\mathfrak{H}$ is a subdivision of $\mathcal{C}^\mathfrak{H}$. For a foil L , also denote $L_+ := L^\mathfrak{H}(\mathfrak{H}(X))$ for any $X \in L$. Note that if $X, X' \in L$ then $L^\mathfrak{H}(\mathfrak{H}(X)) = L^\mathfrak{H}(\mathfrak{H}(X'))$ so L_+ is well-defined. If there is $Y \in \mathbf{m}$ such that $\mathfrak{H}(Y) \in L$, then set $L_- := L^\mathfrak{H}(Y)$. Then L_- is well-defined because if Y, Y' are both such that $\mathfrak{H}(Y), \mathfrak{H}(Y') \in L$, then $L(Y) = L(Y')$. It holds that $(L_+)_- = L$ and when L_- exists $(L_-)_+ = L$.

Example 2.1.4 (Snap-to-grid Shift on Circle Group Revisited). See Figure 2 for an example component, foil, and trajectory determined by the snap-to-grid point-shift of Section 1.3.

It will be important later to know that the graph $G^\mathfrak{H}$ is locally finite. The following result, generalizing one in [2], guarantees this. It crucially relies on the unimodularity of \mathbb{X} .

Proposition 2.1.5. Let $D_n(X)$ denote the n -th order descendants of X , i.e. $D_n(X) := \{Y \in \mathbf{m} : \mathfrak{H}^n(Y) = X\}$. Also let $D(X) := \bigcup_{n=1}^\infty D_n(X)$. Then letting $d_n(X) := \text{card}(D_n(X))$, $d(X) := \text{card}(D(X))$, one has for every $n \geq 0$ that $\mathbf{E}^\mathbf{m}[d_n(0)] = 1$. In particular $d_n(0)$ is $\mathbf{P}^\mathbf{m}$ -a.s. finite, or equivalently \mathbf{P} -a.s. every $X \in \mathbf{m}$ has $d_n(X)$ finite. If in addition, $G^\mathfrak{H}$ is $\mathbf{P}^\mathbf{m}$ -a.s. acyclic, then $\mathbf{E}^\mathbf{m}[d(0)] = \infty$.

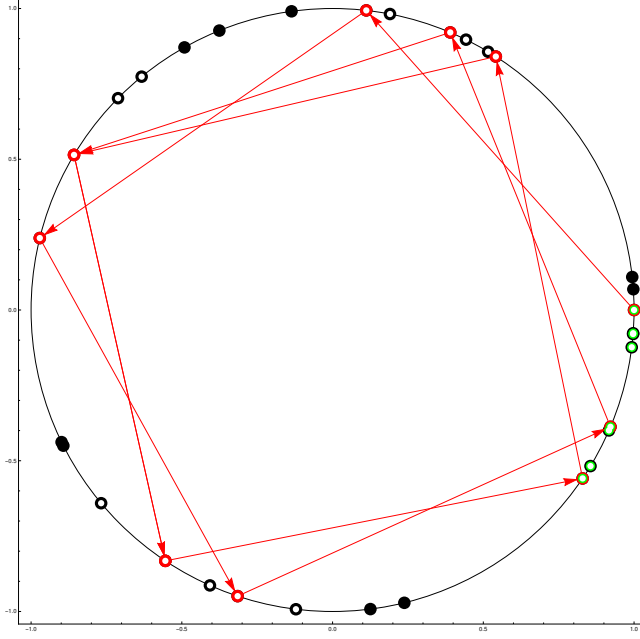


Figure 2: Snap-to-grid point-shift with $z = i$: iterates of $e = 1$ are shown in red, and white and green dots inside a dot indicate membership in the component and foil of e respectively.

Proof. \mathfrak{H}^n is a point-shift in its own right, so the mass transport theorem implies $\mathbf{E}^{\mathfrak{m}}[d_n(0)] = \mathbf{E}^{\mathfrak{m}}[\text{card}(D_n(0))] = 1$ since \mathbb{X} is unimodular. Thus $d_n(0) < \infty$, $\mathbf{P}^{\mathfrak{m}}$ -a.s., and hence \mathbf{P} -a.s. $d_n(X) < \infty$ for all $X \in \mathfrak{m}$ by Theorem 1.2.3. Moreover, when $G^{\mathfrak{H}}$ is acyclic, the D_n partition D and hence $\mathbf{E}^{\mathfrak{m}}[d(0)] = \sum_{n=1}^{\infty} \mathbf{E}^{\mathfrak{m}}[d_n(0)] = \infty$. \square

The primary tool needed to prove the classification theorem follows. It says that it is not possible to extract finite subsets of infinite subsets of \mathfrak{m} in a flow-adapted way. An equivalent result for unimodular networks is Lemma 3.23 in [3] and the proof there is adapted for use here.

Theorem 2.1.6. *Let N be an $\mathbb{N} \cup \{\infty\}$ -valued flow-invariant random variable and let $\mathfrak{N} = \{\mathfrak{N}_i\}_{1 \leq i \leq N}$ be a flow-adapted collection of infinite measurable subsets of \mathfrak{m} and let k be the number of i such that $e \in \mathfrak{N}_i$. Suppose that $\mathbf{E}^{\mathfrak{m}}[k] < \infty$. If \mathfrak{n} is a measurable flow-adapted subset of \mathfrak{m} for which \mathbf{P} -a.s. $\text{card}(\mathfrak{n} \cap \mathfrak{N}_i) < \infty$ for each i , then \mathbf{P} -a.s. $\mathfrak{n} \cap \mathfrak{N}_i = \emptyset$ for all i . In particular, if $\mathfrak{n} \subseteq \bigcup \mathfrak{N}$, then \mathbf{P} -a.s. $\mathfrak{n} = \emptyset$.*

Proof. Define

$$\tau(\omega, x, y) := \sum_{i=1}^{N(\omega)} 1_{x, y \in \mathfrak{N}_i(\omega), y \in \mathfrak{n}(\omega)} \frac{1}{\text{card}(\mathfrak{n}(\omega) \cap \mathfrak{N}_i(\omega))}.$$

The assumptions about flow-adaptedness of \mathfrak{N} , \mathfrak{n} , and \mathfrak{m} , and the flow-invariance of N imply that τ is diagonally invariant. Then $\int_{\mathbb{X}} \tau(e, y) \mathfrak{m}(dy) = k$ by construction since e is in k of the \mathfrak{N}_i . Also $\int_{\mathbb{X}} \tau(x, e) \mathfrak{m}(dx) = \infty$ if $e \in \mathfrak{n} \cap \mathfrak{N}_i$ for some i because the \mathfrak{N}_i are infinite. But the mass transport theorem implies

$$\mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \tau(x, e) \mathfrak{m}(dx) = \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \tau(e, y) \mathfrak{m}(dy) = \mathbf{E}^{\mathfrak{m}}[k] < \infty,$$

and thus it must be that $\mathbf{P}^{\mathfrak{m}}$ -a.s. $e \notin \mathfrak{n} \cap \mathfrak{N}_i$ for any i . Equivalently, \mathbf{P} -a.s. for all $X \in \mathfrak{m}$ it holds that $X \notin \mathfrak{n} \cap \mathfrak{N}_i$ for any i . Since $\mathfrak{n} \cap \mathfrak{N}_i \subseteq \mathfrak{m}$ for each i , it follows that \mathbf{P} -a.s. $\mathfrak{n} \cap \mathfrak{N}_i = \emptyset$ for all i . \square

More information follows about the structure of the locally finite graph $G^{\mathfrak{H}}$. In particular, cycles in components are unique, infinite components are acyclic, foils in infinite components can be ordered like \mathbb{N} or \mathbb{Z} in a flow-adapted way, and \mathfrak{H} acts bijectively on the primeval elements.

Lemma 2.1.7. *\mathbf{P} -a.s. a connected component C of $G^{\mathfrak{H}}$ is either an infinite tree or has exactly one (directed) cycle $K(C)$ for which for all $Y \in C$ there is $n \in \mathbb{N}$ such that $\mathfrak{H}^n(Y) \in K(C)$. Moreover, \mathbf{P} -a.s. there are no infinite components with a cycle.*

Proof. The fact that all elements in C are connected and have out-degree 1 implies there can be at most one cycle. If there are no cycles then C must be infinite since applying \mathfrak{H} to any element repeatedly must never repeat an element. Otherwise there is one cycle $K(C)$ and connectedness implies for every $Y \in C$ there is $n \in \mathbb{N}$ with $\mathfrak{H}^n(Y) \in K(C)$.

Let \mathfrak{N} be the set of infinite components of $G^{\mathfrak{H}}$ with a cycle, and let $\mathfrak{n} \subseteq \bigcup \mathfrak{N}$ be the union of all the cycles of these components. Since cycles are finite, it follows that $\mathfrak{n} \cap C$ is finite for all components $C \in \mathfrak{N}$. By Theorem 2.1.6 $\mathfrak{n} = \emptyset$ and hence there are no infinite components with a cycle \mathbf{P} -a.s. \square

Definition 2.1.8. Within an infinite acyclic connected component $C \in \mathcal{C}^{\mathfrak{H}}$, it is possible to define an order, called the **foil order**, on the foils $\mathcal{L}^{\mathfrak{H}}(C)$ that are subsets of C . This is accomplished by declaring $L^{\mathfrak{H}}(X) < L_+^{\mathfrak{H}}(X)$ for all $X \in C$. When thinking of $\mathfrak{H}(X)$ as being the father of X , the order is that of seniority.

Lemma 2.1.9. *The foil order on an infinite acyclic component C is a total order on C similar to either the order of \mathbb{Z} or \mathbb{N} .*

Proof. Fix any $X \in C$. Let $L_0 := L^{\mathfrak{H}}(X)$ and recursively define $L_{n+1} := (L_n)_+$ and if it exists $L_{-n-1} := (L_{-n})_-$ for $n > 0$. Let L be a foil in C , then it must be that $L = L_i$ for some i . Indeed, let $Y \in L$ and by definition of connectedness choose n, m such that $\mathfrak{H}^n(Y) = \mathfrak{H}^m(X) \in L_m$. It then follows by induction that $Y \in L_{m-n}$, and hence $L^{\mathfrak{H}}(Y) = L_{m-n}$. Next it is shown that $i \mapsto L_i$ is injective. Suppose for contradiction that $L_j = L_{j+N}$. Then there are N pairs (X_i, Y_{i+1}) with $X_i \in L_i, Y_{i+1} \in L_{i+1}$ such that $\mathfrak{H}(X_i) = Y_{i+1}$ for $j \leq i \leq j+N-1$. Since $L_j = L_{j+N}$ it follows that $X_j, Y_{j+N} \in L_j$. Hence it is possible to choose n such

that $\mathfrak{H}^n(X_j) = \mathfrak{H}^n(Y_{j+N})$ and $\mathfrak{H}^n(X_i) = \mathfrak{H}^n(Y_i)$ for all $j+1 \leq i \leq j+N-1$. Then

$$\begin{aligned} \mathfrak{H}^N(\mathfrak{H}^n(X_j)) &= \mathfrak{H}^{N-1}(\mathfrak{H}^n(Y_{j+1})) \\ &= \mathfrak{H}^{N-1}(\mathfrak{H}^n(X_{j+1})) \\ &= \dots \\ &= \mathfrak{H}^0(\mathfrak{H}^n(Y_{j+N})) \\ &= \mathfrak{H}^n(X_j) \end{aligned}$$

contradicts that C is acyclic. Thus $i \mapsto L_i$ is injective. If there is a smallest foil L_{i_0} then $i \mapsto L_{i_0+i}$ is an order isomorphism with \mathbb{N} , otherwise $i \mapsto L_i$ is an order isomorphism with \mathbb{Z} . \square

Lemma 2.1.10. *\mathbf{P} -a.s. \mathfrak{H} restricts to a bijective point-shift $\mathfrak{H}|_{\mathfrak{n}}$ on the flow-adapted sub-process $\mathfrak{H}^\infty(\mathfrak{m})$ of primeval elements.*

Proof. To emphasize that $\mathfrak{H}^\infty(\mathfrak{m})$ is a sub-process of \mathfrak{m} , let $\mathfrak{n} := \mathfrak{H}^\infty(\mathfrak{m})$. It was already noted that \mathfrak{n} is a flow-adapted, and \mathfrak{H} naturally restricts to a point-shift $\mathfrak{H}|_{\mathfrak{n}}$ on \mathfrak{n} because if $X \in \mathfrak{H}^\infty(\mathfrak{m})$ then $\mathfrak{H}(X) \in \mathfrak{H}^\infty(\mathfrak{m})$. By definition, primeval elements are in the image $\mathfrak{H}(\mathfrak{m})$, but moreover they are in the image $\mathfrak{H}(\mathfrak{n})$. Indeed, by Proposition 2.1.5 points in \mathfrak{m} have only finitely many children. If $X \in \mathfrak{n}$ were such that none of its children were primeval, then there would be $n \in \mathbb{N}$ large enough that none of X 's children are in the image $\mathfrak{H}^n(\mathfrak{m})$. But then X would not be in $\mathfrak{H}^{n+1}(\mathfrak{m})$, contradicting that $X \in \mathfrak{H}^\infty(\mathfrak{m})$. Thus the restricted point-shift $\mathfrak{H}|_{\mathfrak{n}}$ is surjective. If \mathfrak{n} is not the empty process \mathbf{P} -a.s. then it has nonzero and finite intensity and $\mathbf{E}^{\mathfrak{n}}[\Delta(\mathfrak{h}^{-1})] = 1$ by unimodularity so that surjectivity and injectivity are equivalent by Proposition 1.2.5 (j), so $\mathfrak{H}|_{\mathfrak{n}}$ is bijective. \square

The main result of this section follows.

Theorem 2.1.11 (Cardinality Classification of a Component). *\mathbf{P} -a.s. each connected component C of $G^{\mathfrak{H}}$ is in one of the three following classes:*

1. **Class \mathcal{F}/\mathcal{F} :** *C is finite, and hence so is each of its \mathfrak{H} -foils. In this case, when denoting by $1 \leq n = n(C) < \infty$ the number of its foils:*
 - C has a unique cycle of length n ;
 - $\mathfrak{H}^\infty(\mathfrak{m}) \cap C$ is the set of vertices of this cycle.
2. **Class \mathcal{I}/\mathcal{F} :** *C is infinite and each of its \mathfrak{H} -foils is finite. In this case:*
 - C is acyclic;
 - Each foil has a junior foil;
 - $\mathfrak{H}^\infty(\mathfrak{m}) \cap C$ is a unique **bi-infinite** path, i.e. a sequence $\{X_n\}_{n \in \mathbb{Z}}$ of points of \mathfrak{m} such that $\mathfrak{H}(X_n) = X_{n+1}$ for all n .

3. **Class \mathcal{I}/\mathcal{I} :** C is infinite and all its \mathfrak{H} -foils are infinite. In this case:

- C is acyclic;
- $\mathfrak{H}^\infty(\mathfrak{m}) \cap C = \emptyset$.

Proof. The properties of finite components C are immediate, so only infinite components are considered. Recall that by Lemma 2.1.7 **P**-a.s. all infinite components are acyclic. Consider the collection \mathfrak{N} of all infinite components that have both finite and infinite foils. Suppose $C \in \mathfrak{N}$. According to Proposition 2.1.5, all $X \in \mathfrak{m}$ have only finitely many children, so that if L is an infinite foil, then L_+ is also infinite. It follows that there is a maximum finite foil L with respect to the foil order in C . Let $\mathfrak{n} \subseteq \bigcup \mathfrak{N}$ be the union of these maximum finite foils of each $C \in \mathfrak{N}$. By construction, $\mathfrak{n} \cap C$ is finite for each $C \in \mathfrak{N}$, so Theorem 2.1.6 implies $\mathfrak{n} = \emptyset$ and hence $\mathfrak{N} = \emptyset$, **P**-a.s. Thus **P**-a.s. each infinite component is either of class \mathcal{I}/\mathcal{F} or \mathcal{I}/\mathcal{I} .

Next, redefine \mathfrak{N} to be the set of infinite foils L of \mathfrak{m} , and let $\mathfrak{n} := \mathfrak{H}^\infty(\mathfrak{m})$. By construction $\mathfrak{n} \cap L$ is finite for each $L \in \mathfrak{N}$ because a foil cannot have multiple primeval elements. If $X \neq Y \in L$ were both primeval, then with n minimal such that $\mathfrak{H}^n(X) = \mathfrak{H}^n(Y)$ one finds the primeval element $\mathfrak{H}^n(X)$ is the image of two distinct primeval elements $\mathfrak{H}^{n-1}(X), \mathfrak{H}^{n-1}(Y)$, contradicting injectivity of $\mathfrak{H}|_{\mathfrak{n}}$ guaranteed by Lemma 2.1.10. Thus Theorem 2.1.6 implies **P**-a.s. $\mathfrak{n} \cap L = \emptyset$ for all infinite foils L , and hence **P**-a.s. $\mathfrak{H}^\infty(\mathfrak{m}) \cap C \neq \emptyset$ implies C is of class \mathcal{I}/\mathcal{F} for all components C .

Conversely, it will be shown that if C is class \mathcal{I}/\mathcal{F} , then $\mathfrak{H}^\infty(\mathfrak{m}) \cap C \neq \emptyset$. Indeed, first consider the collection \mathfrak{N} of components C of class \mathcal{I}/\mathcal{F} that have a minimum foil in the foil order. Letting $\mathfrak{n} \subseteq \bigcup \mathfrak{N}$ be the union of minimum foils in C , it holds that $\mathfrak{n} \cap C$ is the (finite) minimum foil in C for each $C \in \mathfrak{N}$. Thus Theorem 2.1.6 implies $\mathfrak{n} = \emptyset$ and hence $\mathfrak{N} = \emptyset$, **P**-a.s. Now consider a C of class \mathcal{I}/\mathcal{F} and an arbitrary foil L of C . Since L is finite there is a minimum n such that $\mathfrak{H}^n(L)$ is a single point. Let C_0 denote the subgraph of $G^\mathfrak{H}$ of L together with all descendants of elements of L and all forefathers of elements of L up to $\mathfrak{H}^n(L)$. Then C_0 is an infinite connected graph with vertices of finite degree, and hence it contains an infinite simple path $\{X_i\}_{i \leq 0}$ with $\mathfrak{H}(X_i) = X_{i+1}$ for each $i < 0$ by König's infinity lemma (c.f. Theorem 6 in [6]). For $i > 0$, define $X_i := \mathfrak{H}^i(X_0)$. Then $\{X_i\}_{i \in \mathbb{Z}}$ is a bi-infinite path in C satisfying $\mathfrak{H}(X_i) = X_{i+1}$ for all $i \in \mathbb{Z}$, and thus $\{X_i\}_{i \in \mathbb{Z}} \subseteq \mathfrak{H}^\infty(\mathfrak{m}) \cap C$, in particular showing $\mathfrak{H}^\infty(\mathfrak{m}) \cap C \neq \emptyset$. It also holds that $\mathfrak{H}^\infty(\mathfrak{m}) \cap C \subseteq \{X_i\}_{i \in \mathbb{Z}}$ since for any $X \in \mathfrak{H}^\infty(\mathfrak{m}) \cap C$ it is possible to choose n, m such that $\mathfrak{H}^n(X) = \mathfrak{H}^m(X_0) = X_m$. Uniqueness of primeval children then implies $X = X_{m-n}$. It follows that $\mathfrak{H}^\infty(\mathfrak{m}) \cap C = \{X_i\}_{i \in \mathbb{Z}}$.

Thus it is shown that **P**-a.s. infinite components C are class \mathcal{I}/\mathcal{F} iff $\mathfrak{H}^\infty(\mathfrak{m}) \cap C \neq \emptyset$ and in this case $\mathfrak{H}^\infty(\mathfrak{m}) \cap C$ is a unique bi-infinite sequence $\{X_i\}_{i \in \mathbb{Z}}$ satisfying $\mathfrak{H}(X_i) = X_{i+1}$. Since \mathcal{I}/\mathcal{F} and \mathcal{I}/\mathcal{I} are the only possible choices, by process of elimination it follows that **P**-a.s. infinite components C are of class \mathcal{I}/\mathcal{I} iff $\mathfrak{H}^\infty(\mathfrak{m}) \cap C = \emptyset$. \square

2.2. A Counterexample on a Non-unimodular Group

(Strip Point-shift on $ax+b$ Group Revisited). This example serves to show that the cardinality classification (Theorem 2.1.11) does not hold for non-unimodular spaces. It is an open question whether a more general classification for such spaces exists. Consider again the strip point-shift of Section 1.3. Suppose that $2\delta\gamma < 1$. It will be shown that \mathbf{P} -a.s. $\mathfrak{H}^n(X)$ eventually becomes constant as $n \rightarrow \infty$ for all $X \in \mathfrak{m}$. It suffices to show that under $\mathbf{P}^{\mathfrak{m}}$ it holds that $\mathfrak{H}^n(e)$ eventually becomes constant. Indeed, with $\mu^{(i)}$ denoting the i th factorial moment measure of μ ,

$$\begin{aligned}
& \mathbf{E}^{\mathfrak{m}} \sum_{k=0}^{\infty} \mathfrak{m}(S(\mathfrak{H}^k(e)) \setminus \{\mathfrak{H}^k(e)\}) \\
&= \sum_{k=0}^{\infty} \mathbf{E}^{\mathfrak{m}} \mathfrak{m}(S(\mathfrak{H}^k(e)) \setminus \{\mathfrak{H}^k(e)\}) \\
&\leq \sum_{k=0}^{\infty} \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}^{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} (\mathfrak{m}^1)^{(k+1)}(dx_1 \times \cdots \times dx_{k+1}) \\
&= \sum_{k=0}^{\infty} \mathbf{E} \int_{\mathbb{X}^{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} \mathfrak{m}^{(k+1)}(dx_1 \times \cdots \times dx_{k+1}) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{X}^{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} \gamma^{k+1} \lambda(dx_{k+1}) \cdots \lambda(dx_1) \\
&= \sum_{k=0}^{\infty} (2\delta)^{k+1} \gamma^{k+1} \\
&< \infty,
\end{aligned}$$

where here the Slivnyak-Mecke theorem is used along with the fact that the factorial moment measures of a Poisson point process are just powers of the intensity measure. Thus it must be that $\mathfrak{m}(S(\mathfrak{H}^k(e)) \setminus \{\mathfrak{H}^k(e)\}) = 0$ for all k large, $\mathbf{P}^{\mathfrak{m}}$ -a.s. That is, there are no points of \mathfrak{m} in $S(\mathfrak{H}^k(e))$ besides $\mathfrak{H}^k(e)$ itself. Consequently, $\mathfrak{H}^k(e)$ is a fixed point of \mathfrak{H} and $\mathfrak{H}^k(e)$ is thus eventually constant in k . Equivalently, \mathbf{P} -a.s. for every $X \in \mathfrak{m}$ it holds that $\mathfrak{H}^k(X)$ is eventually constant in k .

Next it will be shown that every fixed point of \mathfrak{H} is the image of infinitely many $X \in \mathfrak{m}$. Again it is enough to show under $\mathbf{P}^{\mathfrak{m}}$ that if $\mathfrak{H}(e) = e$ then e is the image of infinitely many $X \in \mathfrak{m}$. This is accomplished by finding a region of points $(x, y) \in \mathbb{X}$ such that

- (i) $(1, 0) \in S(x, y)$, and
- (ii) $S(x, y) \cap ([1, \infty) \times \mathbb{R}) \subseteq S(1, 0)$,

which implies \mathfrak{H} would map a point of \mathfrak{m} at (x, y) to $(1, 0)$. The condition (i) says $1 \geq x$ and $y - \delta x \leq 0 \leq y + \delta x$, i.e. $-\delta x \leq y \leq \delta x$. Condition (ii) is

guaranteed if $[y - \delta x, y + \delta x] \subseteq [-\delta, \delta]$, i.e. if $y \geq \delta(x - 1)$ and $y \leq \delta(1 - x)$. The constraints

$$0 < x \leq 1, \quad -\delta x \leq y \leq \delta x, \quad y \leq \delta(1 - x), \quad y \geq \delta(x - 1),$$

bound a parallelogram D with corners

$$(0, 0), \quad (1/2, \delta/2), \quad (1, 0), \quad (1/2, -\delta/2).$$

Then

$$\mathbf{E}^{\mathbf{m}}[\mathbf{m}^{\dagger}(D)] = \gamma \lambda(D) \geq \gamma \int_0^{1/2} \int_{-\delta x}^{\delta x} \frac{1}{x^2} dy dx = \gamma \int_0^{1/2} \frac{2\delta}{x} dx = \infty$$

so that the region D contains infinitely many points of \mathbf{m} , $\mathbf{P}^{\mathbf{m}}$ -a.s. By construction, if $\mathfrak{H}(e) = e$ then every $X \in \mathbf{m} \cap D$ has $\mathfrak{H}(X) = e$, proving the claim.

Putting previous claims together, it holds that the foils and connected components are identical because every component contains a fixed point, and the foils and components are in bijection with the fixed points of \mathfrak{H} . The connected component of a fixed point Y of \mathfrak{H} is all $X \in \mathbf{m}$ that are eventually sent to Y . Thus all components and foils are infinite (class \mathcal{I}/\mathcal{I}). However, the components are not acyclic and $\mathfrak{H}^{\infty}(\mathbf{m}) = \{X \in \mathbf{m} : \mathfrak{H}(X) = X\} \neq \emptyset$, contrary to what the classification theorem would suggest for unimodular \mathbb{X} . It follows that the properties of the cardinality classification cannot be extended beyond the case of unimodular \mathbb{X} .

3. Properties of Point-shifts

3.1. Mecke's Invariance Theorem

In the case of $\mathbb{X} = \mathbb{R}^d$ Mecke's invariance theorem shows that Palm probabilities are preserved under bijective point-shifts. It will be shown in Corollary 3.1.1 that if \mathbb{X} is unimodular then this still holds. Even stronger, a point-shift is bijective iff it preserves Palm probabilities. However, for non-unimodular \mathbb{X} this is not so. Precisely, a notion of isomodularity will be defined and it will be proved that amongst bijective point-shifts isomodular ones are exactly those that preserve Palm probabilities (Theorem 3.1.6).

For the rest of the section, fix a stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbf{P})$, a compatible simple point process \mathbf{m} of intensity $\gamma \in (0, \infty)$, and a point-map \mathfrak{h} with associated point-shift \mathfrak{H} . The notation for the corresponding notation $\tau^{\mathfrak{H}}$, h^+ , h^- , H^+ , H^- , \mathfrak{h}^- , \mathfrak{H}^- mentioned in the preliminaries is retained.

The simple case of Mecke's invariance theorem when \mathbb{X} is unimodular follows.

Corollary 3.1.1 (Mecke's Invariance Theorem). *Suppose that \mathbb{X} is unimodular. Then \mathfrak{H} preserves $\mathbf{P}^{\mathbf{m}}$ iff \mathfrak{H} is bijective. That is, $\mathbf{P}^{\mathbf{m}}(\theta_{\mathfrak{h}}^{-1} \in A) = \mathbf{P}^{\mathbf{m}}(A)$ for all $A \in \mathcal{A}$ iff \mathfrak{H} is bijective.*

Proof. Apply Proposition 1.2.5 (f), the test for bijectivity, and use the fact that $\Delta(x) = 1$ for all $x \in \mathbb{X}$. \square

With Mecke's invariance theorem for unimodular \mathbb{X} in place, one may ask about non-unimodular \mathbb{X} . For these \mathbb{X} , which bijective point-shifts preserve Palm probabilities? Equation (4) shows that the obstruction is the factor $\Delta(\mathfrak{h}^{-1})$. This motivates the definition of isomodularity, which says that a point-shift preserves the value of $\Delta(X)$ for each $X \in \mathfrak{m}$. Isomodularity is a special case of invariance of a subgroup under \mathfrak{H} , which is defined presently.

Definition 3.1.2. A measurable subgroup $G \in \mathcal{B}(\mathbb{X})$ of \mathbb{X} is called **\mathfrak{H} -invariant** if \mathbf{P} -a.s. $\mathfrak{H}(X)$ is in the same coset as X for all $X \in \mathfrak{m}$. If $\{\Delta = 1\}$ is \mathfrak{H} -invariant, i.e. if \mathbf{P} -a.s. $\Delta(\mathfrak{H}(X)) = \Delta(X)$ for all $X \in \mathfrak{m}$, one calls \mathfrak{H} **isomodular**.

Lemma 3.1.3. *If \mathbb{X} is unimodular, then \mathfrak{H} is isomodular.*

Proof. If \mathbb{X} is unimodular, then all point-shifts \mathfrak{H} are isomodular because $\Delta(x) = 1$ for all $x \in \mathbb{X}$. \square

A brief detour is taken to go through the equivalent descriptions of \mathfrak{H} -invariance under \mathbf{P} and $\mathbf{P}^{\mathfrak{m}}$.

Proposition 3.1.4. *Let $G \in \mathcal{B}(\mathbb{X})$ a measurable subgroup of \mathbb{X} , and for each $x \in \mathbb{X}$ let $[x] := xG$ denote the coset of x . Then the following are equivalent*

- (a) G is \mathfrak{H} -invariant, i.e. \mathbf{P} -a.s. $[\mathfrak{H}(X)] = [X]$ for all $X \in \mathfrak{m}$,
- (b) $\mathbf{P}^{\mathfrak{m}}$ -a.s. $[\mathfrak{h}] = [e]$,

and if \mathfrak{H} is bijective, the previous statements are also equivalent to

- (c) \mathbf{P} -a.s. $[\mathfrak{H}^{-}(X)] = [X]$ for all $X \in \mathfrak{m}$,
- (d) $\mathbf{P}^{\mathfrak{m}}$ -a.s. $[\mathfrak{h}^{-}] = [e]$.

Proof.

(a) \iff (b): The equivalence follows from Theorem 1.2.3, so that $\mathbf{P}^{\mathfrak{m}}$ -a.s. $[\mathfrak{h}] = [e]$ is equivalent to \mathbf{P} -a.s. $[\mathfrak{h}(\theta_X^{-1})] = [e]$ for all $X \in \mathfrak{m}$, which is the same as $[\mathfrak{H}(X)] = [X]$ after multiplying by X .

(a) \iff (c): Using that \mathfrak{H} and \mathfrak{H}^{-} are inverses, replace X with $\mathfrak{H}^{-}(X)$ in (b) to get (c) or replace X with $\mathfrak{H}(X)$ in (c) to get (b).

(c) \iff (d): Same proof as (a) \iff (b). \square

Since isomodularity plays an important role in what follows, the previous result is restated for $G := \{\Delta = 1\}$.

Corollary 3.1.5. *Let \mathfrak{H} be bijective, then the following are equivalent*

- (a) \mathfrak{H} is isomodular, i.e. \mathbf{P} -a.s. $\Delta(\mathfrak{H}(X)) = \Delta(X)$ for all $X \in \mathfrak{m}$,
- (b) $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\Delta(\mathfrak{h}) = 1$,
- (c) \mathbf{P} -a.s. $\Delta(\mathfrak{H}^-(X)) = \Delta(X)$ for all $X \in \mathfrak{m}$,
- (d) $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\Delta(\mathfrak{h}^-) = 1$.

Now the question of which bijective point-shifts preserve Palm probabilities is answerable.

Theorem 3.1.6. *Suppose \mathfrak{H} is bijective. Then \mathfrak{H} preserves $\mathbf{P}^{\mathfrak{m}}$ iff \mathfrak{H} is isomodular. That is, $\mathbf{P}^{\mathfrak{m}}(\theta_{\mathfrak{h}}^{-1} \in A) = \mathbf{P}^{\mathfrak{m}}(A)$ for all $A \in \mathcal{A}$ iff \mathfrak{H} is isomodular.*

Proof. Suppose \mathfrak{H} is isomodular. Then $\Delta(\mathfrak{h}^-) = 1$, $\mathbf{P}^{\mathfrak{m}}$ -a.s. by Corollary 3.1.5. Hence (5) immediately implies \mathfrak{H} preserves $\mathbf{P}^{\mathfrak{m}}$. If \mathfrak{H} not isomodular, at least one of $\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^-) > 1)$ and $\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^-) < 1)$ is strictly positive. The cases are nearly identical, so assume $\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^-) > 1) > 0$ and take $A := \{\Delta(\mathfrak{h}^-) > 1\}$. Then take $f := 1_A$ in (5) to find

$$\mathbf{P}^{\mathfrak{m}}(\theta_{\mathfrak{h}}^{-1} \in A) = \mathbf{E}^{\mathfrak{m}} \left[\frac{1_{\Delta(\mathfrak{h}^-) > 1}}{\Delta(\mathfrak{h}^-)} \right] < \mathbf{E}^{\mathfrak{m}} [1_{\Delta(\mathfrak{h}^-) > 1}] = \mathbf{P}^{\mathfrak{m}}(A),$$

showing that $\mathbf{P}^{\mathfrak{m}}$ is not preserved. \square

3.2. Reciprocal and Reverse of a Point-map

A curious interplay between the reverse \mathfrak{h}^- and the reciprocal \mathfrak{h}^{-1} of a point-map is investigated, and a characterization of when the two have the same law under $\mathbf{P}^{\mathfrak{m}}$ is given.

The notation of the previous section is retained. That is, $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbf{P})$ is a stationary framework, \mathfrak{m} is a compatible simple point process of intensity $\gamma \in (0, \infty)$, and \mathfrak{h} is a point-map with associated point-shift \mathfrak{H} . The notation for the corresponding $\tau^{\mathfrak{H}}$, h^+ , h^- , H^+ , H^- , \mathfrak{h}^- , \mathfrak{H}^- is also retained. Next follows another result along the lines of Proposition 1.2.5 (f) and (g) which sparks interest in the distributional relationship between \mathfrak{h}^{-1} and \mathfrak{h}^- .

Corollary 3.2.1. *Suppose \mathfrak{H} is bijective. For all $f : \mathbb{X} \rightarrow \mathbb{R}_+$ measurable it holds that*

$$\mathbf{E}^{\mathfrak{m}} [f(\mathfrak{h}^{-1})\Delta(\mathfrak{h}^{-1})] = \mathbf{E}^{\mathfrak{m}} [f(\mathfrak{h}^-)], \quad (6)$$

$$\mathbf{E}^{\mathfrak{m}} [f(\mathfrak{h}^{-1})] = \mathbf{E}^{\mathfrak{m}} \left[\frac{f(\mathfrak{h}^-)}{\Delta(\mathfrak{h}^-)} \right]. \quad (7)$$

Proof. Use the fact that $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\mathfrak{h}^-(\theta_{\mathfrak{h}}^{-1}) = \mathfrak{h}^{-1}\mathfrak{H}^-(\mathfrak{H}(e)) = \mathfrak{h}^{-1}$ and replace f by $f(\mathfrak{h}^-)$ in each of (3) and (5). \square

One sees in (7) that non-unimodularity of \mathbb{X} is, as usual, an obstruction. Two more results relating the distributions of $\Delta(\mathfrak{h}^-)$ and $\Delta(\mathfrak{h}^{-1})$ are given. Then it is shown in Theorem 3.2.4 that amongst bijective point-shifts, the isomodular ones are precisely those for which \mathfrak{h}^{-1} and \mathfrak{h}^- have the same distribution under $\mathbf{P}^{\mathfrak{m}}$. Recall that this was also the class of point-shifts that preserve Palm probabilities by Theorem 3.1.6.

Corollary 3.2.2. *Let \mathfrak{H} be bijective, then for all $r > 0$ it holds that*

$$r\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^{-1}) = r) = \mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^-) = r),$$

and if this number is strictly positive then for all $A \in \mathcal{A}$

$$\mathbf{P}^{\mathfrak{m}}(\theta_{\mathfrak{h}}^{-1} \in A \mid \Delta(\mathfrak{h}^{-1}) = r) = \mathbf{P}^{\mathfrak{m}}(A \mid \Delta(\mathfrak{h}^-) = r).$$

Proof. Fix $r > 0$ and take $f(x) := 1_{\Delta(x)=r}$ in (7). One finds

$$\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^{-1}) = r) = \frac{1}{r}\mathbf{P}^{\mathfrak{m}}(\Delta(\mathfrak{h}^-) = r) =: p$$

showing the first claim. Supposing that $p > 0$, take $f := 1_A 1_{\Delta(\mathfrak{h}^-)=r}$ in (5) and use that $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\mathfrak{h}^-(\theta_{\mathfrak{h}}^{-1}) = \mathfrak{h}^{-1}$ to find

$$\mathbf{P}^{\mathfrak{m}}(\theta_{\mathfrak{h}}^{-1} \in A, \Delta(\mathfrak{h}^{-1}) = r) = \frac{1}{r}\mathbf{P}^{\mathfrak{m}}(A, \Delta(\mathfrak{h}^-) = r).$$

Division by p finishes the proof. \square

Lemma 3.2.3. *Let \mathfrak{H} be bijective, then for all $\alpha \in \mathbb{R}$ and $0 \leq r \leq s \leq \infty$ it holds that*

$$\mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})^{\alpha} 1_{r \leq \Delta(\mathfrak{h}^{-1}) \leq s}] = \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^-)^{\alpha-1} 1_{r \leq \Delta(\mathfrak{h}^-) \leq s}]. \quad (8)$$

Proof. Take $f(x) := \Delta(x)^{\alpha} 1_{r \leq \Delta(x) \leq s}$ in (7). \square

Theorem 3.2.4. *Let \mathfrak{H} be bijective, then \mathfrak{h}^{-1} and \mathfrak{h}^- have the same law under $\mathbf{P}^{\mathfrak{m}}$ iff \mathfrak{H} is isomodular.*

Proof. Suppose \mathfrak{H} is isomodular. Then by Corollary 3.1.5, $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\Delta(\mathfrak{h}) = \Delta(\mathfrak{h}^-) = 1$ and thus (7) shows that \mathfrak{h}^{-1} and \mathfrak{h}^- have the same law under $\mathbf{P}^{\mathfrak{m}}$.

Next suppose that \mathfrak{h}^{-1} and \mathfrak{h}^- have the same law under $\mathbf{P}^{\mathfrak{m}}$. Then

$$\mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})^{\alpha} 1_{r \leq \Delta(\mathfrak{h}^{-1}) \leq s}] = \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^-)^{\alpha} 1_{r \leq \Delta(\mathfrak{h}^-) \leq s}] \quad (9)$$

for all $\alpha \in \mathbb{R}$ and all $0 \leq r \leq s \leq \infty$. But then for all $\alpha \in \mathbb{R}$ and all $0 \leq r \leq s \leq \infty$

$$\begin{aligned} \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})^{\alpha+1} 1_{r \leq \Delta(\mathfrak{h}^{-1}) \leq s}] &= \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^-)^{\alpha} 1_{r \leq \Delta(\mathfrak{h}^-) \leq s}] && \text{(by (8))} \\ &= \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^{-1})^{\alpha} 1_{r \leq \Delta(\mathfrak{h}^{-1}) \leq s}] && \text{(by (9))} \\ &= \mathbf{E}^{\mathfrak{m}}[\Delta(\mathfrak{h}^-)^{\alpha-1} 1_{r \leq \Delta(\mathfrak{h}^{-1}) \leq s}]. && \text{(by (8))} \end{aligned}$$

Taking $\alpha := 1, r := 1, s := \infty$

$$\mathbf{E}^{\mathbf{m}}[\Delta(\mathfrak{h}^{-1})^2 1_{1 \leq \Delta(\mathfrak{h}^{-1})}] = \mathbf{E}^{\mathbf{m}}[\Delta(\mathfrak{h}^{-1}) 1_{1 \leq \Delta(\mathfrak{h}^{-1})}]$$

which is absurd unless $\Delta(\mathfrak{h}^{-1}) \leq 1$, $\mathbf{P}^{\mathbf{m}}$ -a.s. It also holds that with $\alpha := 1, r := 0, s := 1$,

$$\mathbf{E}^{\mathbf{m}}[\Delta(\mathfrak{h}^{-1})^2 1_{\Delta(\mathfrak{h}^{-1}) \leq 1}] = \mathbf{E}^{\mathbf{m}}[\Delta(\mathfrak{h}^{-1}) 1_{\Delta(\mathfrak{h}^{-1}) \leq 1}],$$

which is absurd unless $\Delta(\mathfrak{h}^{-1}) \geq 1$, $\mathbf{P}^{\mathbf{m}}$ -a.s. It follows that $\Delta(\mathfrak{h}^{-1}) = 1$, $\mathbf{P}^{\mathbf{m}}$ -a.s. By Corollary 3.1.5 the result follows. \square

3.3. Separating Points of a Point Process

In this section a notion of a function separating points of a point process is introduced. As always, \mathbf{m} is a simple and compatible point process of intensity $\gamma \in (0, \infty)$ on a stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbf{P})$.

Definition 3.3.1. Let S be a set, $f : \mathbb{X} \rightarrow S$, and suppose that \mathbf{P} -a.s. no distinct $X, Y \in \mathbf{m}$ have $f(X) = f(Y)$. Then say that f **separates points** of \mathbf{m} . Similarly, say that a fixed partition $\{B_i\}_{i \in J}$ of \mathbb{X} **separates points** of \mathbf{m} if \mathbf{P} -a.s. no B_i contains more than 1 point of \mathbf{m} .

When separation of points occurs is studied by proving a general result concerning when there cannot be an n -tuple of distinct points of \mathbf{m} satisfying a given property. Recall that the n -th **factorial moment measure** of a counting measure μ with representation $\mu = \sum_i \delta_{x_i}$ is defined as $\mu^{(n)} := \sum_{i_1 \neq \dots \neq i_n} \delta_{(x_{i_1}, \dots, x_{i_n})}$, where the notation $i_1 \neq \dots \neq i_n$ means that i_1, \dots, i_n are all distinct.

Theorem 3.3.2. Let (S, Σ) be a measurable space and fix $M \in \Sigma$. Let $F : \mathbb{X} \times \mathbb{X}^n \rightarrow S$ be measurable, and suppose that for $\mathbf{E}^{\mathbf{m}}[(\mathbf{m}!)^{(n)}]$ -a.e. $y \in \mathbb{X}^n$,

$$\lambda(x \in \mathbb{X} : F(x, xy) \in M) = 0.$$

Then \mathbf{P} -a.s. no $n+1$ distinct $X, Y_1, \dots, Y_n \in \mathbf{m}$ have $F(X, Y_1, \dots, Y_n) \in M$.

Proof. By straight calculations,

$$\begin{aligned}
& \mathbf{P}(\exists X \in \mathfrak{m}, Y \in \mathfrak{m}^{(n)} : (X, Y) \in \mathfrak{m}^{(n+1)}, F(X, Y) \in M) \\
& \leq \mathbf{E} \int_{\mathbb{X}} 1_{\exists Y \in \mathfrak{m}^{(n)} : \forall i, Y_i \neq x, F(x, Y) \in M} \mathfrak{m}(dx) \\
& \leq \mathbf{E} \int_{\mathbb{X}} \mathfrak{m}^{(n)}(\theta_e, \{y \in \mathbb{X}^n : \forall i, y_i \neq x, F(x, y) \in M\}) \mathfrak{m}(dx) \\
& = \gamma \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \mathfrak{m}^{(n)}(\theta_x, \{y \in \mathbb{X}^n, \forall i, y_i \neq x, F(x, y) \in M\}) \lambda(dx) \\
& = \gamma \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \mathfrak{m}^{(n)}(\theta_e, \{x^{-1}y : y \in \mathbb{X}^n, \forall i, y_i \neq x, F(x, y) \in M\}) \lambda(dx) \\
& = \gamma \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \mathfrak{m}^{(n)}(\theta_e, \{y \in \mathbb{X}^n, \forall i, xy_i \neq x, F(x, xy) \in M\}) \lambda(dx) \\
& = \gamma \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \int_{\mathbb{X}} 1_{F(x, xy) \in M} (\mathfrak{m}^!)^{(n)}(dy) \lambda(dx) \\
& = \gamma \mathbf{E}^{\mathfrak{m}} \int_{\mathbb{X}} \lambda(x \in \mathbb{X} : F(x, xy) \in M) (\mathfrak{m}^!)^{(n)}(dy) \\
& = 0,
\end{aligned}$$

where in the third equality the C-L-M-M theorem is used. This proves the claim. \square

Theorem 3.3.2 immediately gives a condition for separating points of \mathfrak{m} .

Corollary 3.3.3 (Condition for Separating Points). *Let (S, Σ) be a measurable space, $f : \mathbb{X} \rightarrow S$ measurable, and suppose for all $y \neq e$, or more generally for $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!]$ -a.e. $y \in \mathbb{X}$,*

$$\lambda(x \in \mathbb{X} : f(x) = f(xy)) = 0.$$

Then f separates points of \mathfrak{m} . Implicit in the previous line is the assumption that the sets $\{x \in \mathbb{X} : f(x) = f(xy)\}$ are measurable for all $y \in \mathbb{X}$. This is automatic if (S, Σ) is a standard measurable space, or more generally if $S \times S$ has measurable diagonal.

Proof. Take $n := 1$, $F(x, y) := (f(x), f(y))$ for all $x, y \in \mathbb{X}$, and take M to be the diagonal of $S \times S$, then apply Theorem 3.3.2. \square

Note 3.3.4. Corollary 3.3.3 generalizes the well-known theorem in $\mathbb{X} := \mathbb{R}^d$ that a stationary point process has not two points equidistant from 0. That would be the case of $f(x) := |x|$. Not all \mathbb{X} have this property though. Indeed, if \mathbb{X} is a countable group with the discrete distance $d(x, y) := 1_{x \neq y}$, then $\lambda(x \in \mathbb{X} : d(x, e) = d(xy, e)) > 0$ for all $y \neq e$ so the result does not apply if \mathbb{X} has more than one element.

Proposition 3.3.5. *Let $B \in \mathcal{B}(\mathbb{X})$ with $e \in B$. If $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(B)] = 0$, then \mathbf{P} -a.s. for all $X \in \mathfrak{m}$ it holds that $\mathfrak{m}(XB) = 1$, i.e. X is the unique point of \mathfrak{m} inside XB .*

Proof. The hypotheses imply $\mathbf{P}^{\mathfrak{m}}$ -a.s. $\mathfrak{m}(B \setminus \{e\}) = 0$. By Theorem 1.2.3, \mathbf{P} -a.s. all $X \in \mathfrak{m}$ are such that $\theta_X^{-1}\mathfrak{m}(B \setminus \{e\}) = 0$, i.e. $\mathfrak{m}(XB \setminus \{X\}) = 0$, and hence $\mathfrak{m}(XB) = 1$. \square

Corollary 3.3.6. *Let $G \in \mathcal{B}(\mathbb{X})$ a subgroup of \mathbb{X} . If $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(G)] = 0$, then the cosets of G separate points of \mathfrak{m} .* \square

Corollary 3.3.7. *Let $G \in \mathcal{B}(\mathbb{X})$ a subgroup of \mathbb{X} . If $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(G)] = 0$ but G is \mathfrak{H} -invariant for some point-shift \mathfrak{H} , then \mathfrak{H} is the identity point-shift \mathbf{P} -a.s.*

Proof. G being \mathfrak{H} -invariant means $\mathfrak{H}(X)$ and X are in the same coset for $X \in \mathfrak{m}$, then by Corollary 3.3.6 \mathfrak{H} is the identity point-shift. \square

Corollary 3.3.8. *Let $G \in \mathcal{B}(\mathbb{X})$ a subgroup of \mathbb{X} . If $\lambda(G) = 0$ and \mathfrak{m} is Poisson with intensity $\gamma \in (0, \infty)$, then the only \mathfrak{H} for which G is \mathfrak{H} -invariant is the identity.*

Proof. Theorem 4.2.7 implies that $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!] = \gamma\lambda$ so that $\mathbf{E}^{\mathfrak{m}}[\mathfrak{m}^!(G)] = \gamma\lambda(G) = 0$ and Corollary 3.3.7 applies. \square

4. Concluding Remarks and Further Research

4.1. Mecke's Invariance Theorem

In Mecke's invariance theorem for unimodular spaces (Corollary 3.1.1), not only does bijectivity imply \mathfrak{H} preserves Palm probabilities, but the converse holds as well. However, for the non-unimodular case in Theorem 3.1.6, bijectivity is required as an assumption. Are there non-bijective point-shifts that preserve Palm probabilities in non-unimodular spaces?

4.2. Point-shift Foliations

Point-shift foliations are in many ways special cases of vertex-shifts of random networks [3]. A cardinality classification is proved in [3] for unimodular random networks, and questions remain about the relationship of these two frameworks. It is shown in [3] that stationary point processes on \mathbb{R}^d are embeddings of random networks. Talks with the author of [3] suggest that a certain class of unimodular networks, called Eternal Family Trees, can be viewed as stationary point processes on \mathbb{R}^d . Two open questions are whether stationary point processes on unimodular \mathbb{X} can be viewed as unimodular random networks, and whether any unimodular random network can be viewed as a stationary point process on some unimodular \mathbb{X} .

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Appendix: Palm Calculus

Theorem 4.2.1 (Inversion Formula). [7] *There exists a bounded measurable $K : \Omega \times \mathbb{X} \rightarrow \mathbb{R}$ with $K \geq 0$ such that*

$$\int_{\mathbb{X}} K(\theta_e, x) \mathbf{m}(dx) = 1_{\mathbf{m} \neq \mathbf{0}}, \quad (10)$$

and for all $K \geq 0$ (not necessarily bounded) \mathbf{P} -a.s. satisfying (10), it holds that

$$\mathbf{E}[1_{\mathbf{m} \neq \mathbf{0}} f] = \gamma \mathbf{E}^{\mathbf{m}} \int_{\mathbb{X}} f(\theta_x) K(\theta_x, x) \lambda(dx) \quad (11)$$

for all measurable $f : \Omega \rightarrow \mathbb{R}_+$.

Proposition 4.2.2. *If $A \in \mathcal{A}$ is **shift-invariant** in the sense that $A = \theta_x^{-1} A$ for all $x \in \mathbb{X}$, then*

$$\mathbf{P}(A) = 1 \implies \mathbf{P}^{\mathbf{m}}(A) = 1 \implies \mathbf{P}(A \mid \mathbf{m} \neq \mathbf{0}) = 1.$$

In particular, if $\{\mathbf{m} = \mathbf{0}\} \subseteq A$ then

$$\mathbf{P}(A) = 1 \iff \mathbf{P}^{\mathbf{m}}(A) = 1.$$

Proof. Suppose $\mathbf{P}(A) = 1$. From the definition of Palm, for $0 < \lambda(B) < \infty$

$$\begin{aligned} \mathbf{P}^{\mathbf{m}}(A) &= \frac{1}{\gamma \lambda(B)} \mathbf{E} \int_{\mathbb{X}} 1_{x \in B} 1_{\theta_x^{-1} \in A} \mathbf{m}(dx) \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E} \int_{\mathbb{X}} 1_{x \in B} 1_A \mathbf{m}(dx) \quad (\text{shift-invariance of } A) \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E}[1_A \mathbf{m}(B)] \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E}[\mathbf{m}(B)] \quad (\mathbf{P}(A) = 1) \\ &= 1. \end{aligned}$$

Next suppose $\mathbf{P}^{\mathbf{m}}(A) = 1$. Then from Theorem 4.2.1 there is measurable $K : \Omega \times \mathbb{X} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \mathbf{P}(A \cap \{\mathbf{m} \neq \mathbf{0}\}) &= \mathbf{E}[1_{\mathbf{m} \neq \mathbf{0}} 1_A] \\ &= \gamma \mathbf{E}^{\mathbf{m}} \int_{\mathbb{X}} 1_{\theta_x \in A} K(\theta_x, x) \lambda(dx) \quad (\text{inversion formula}) \\ &= \gamma \mathbf{E}^{\mathbf{m}} \left[1_A \int_{\mathbb{X}} K(\theta_x, x) \lambda(dx) \right] \quad (\text{shift-invariance of } A) \\ &= \gamma \mathbf{E}^{\mathbf{m}} \left[\int_{\mathbb{X}} K(\theta_x, x) \lambda(dx) \right] \quad (\mathbf{P}^{\mathbf{m}}(A) = 1) \\ &= \mathbf{E}[1_{\mathbf{m} \neq \mathbf{0}} \cdot 1] \quad (\text{inversion formula}) \\ &= \mathbf{P}(\mathbf{m} \neq \mathbf{0}). \end{aligned}$$

Dividing by $\mathbf{P}(\mathbf{m} \neq \mathbf{0})$ gives $\mathbf{P}(A \mid \mathbf{m} \neq \mathbf{0}) = 1$, and if $\{\mathbf{m} = \mathbf{0}\} \subseteq A$, then

$$\mathbf{P}(A) = \mathbf{P}(A \cap \{\mathbf{m} \neq \mathbf{0}\}) + \mathbf{P}(A \cap \{\mathbf{m} = \mathbf{0}\}) = \mathbf{P}(\mathbf{m} \neq \mathbf{0}) + \mathbf{P}(\mathbf{m} = \mathbf{0}) = 1.$$

□

Corollary 4.2.3. *(\mathbf{m}, \mathbf{P}) is a (simple) point process iff $(\mathbf{m}, \mathbf{P}^{\mathbf{m}})$ is a (simple) point process.*

Proof. The event that $\mathbf{m}(B) \in \mathbb{N} \cup \{\infty\}$ for each $B \in \mathcal{B}(\mathbb{X})$ and the event that $\mathbf{m}(\{x\}) \leq 1$ for all $x \in \mathbb{X}$ are shift-invariant, so Proposition 4.2.2 applies. Equivalence follows because $\mathbf{0}$ is a (simple) counting measure. □

Lemma 4.2.4. *Let $A \in \mathcal{A}$. Then*

$$\mathbf{P}^{\mathbf{m}}(A) = 1 \iff \mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0) = 1.$$

Proof. By replacing A with its complement it is equivalent to show $\mathbf{P}^{\mathbf{m}}(A) = 0$ iff $\mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \in A) > 0) = 0$. Note that it is the joint measurability of the action $(\omega, x) \mapsto \theta_x \omega$ that lets one conclude for $B \in \mathcal{B}(\mathbb{X})$ that sets like

$$\{\mathbf{m}(x \in \mathbb{X} : x \in B, \theta_x^{-1} \in A) > 0\}$$

are measurable.

If $\mathbf{P}^{\mathbf{m}}(A) = 0$, from the definition of Palm, for $0 < \lambda(B) < \infty$,

$$\begin{aligned} 0 &= \mathbf{P}^{\mathbf{m}}(A) \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E} \int_{\mathbb{X}} 1_{x \in B} 1_{\theta_x^{-1} \in A} \mathbf{m}(dx) \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E}[\mathbf{m}(x \in \mathbb{X} : x \in B, \theta_x^{-1} \in A)]. \end{aligned}$$

Thus $\mathbf{E}[\mathbf{m}(x \in \mathbb{X} : x \in B, \theta_x^{-1} \in A)] = 0$ and taking relatively compact B increasing to \mathbb{X} one finds $\mathbf{E}[\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \in A)] = 0$, so $\mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \in A) > 0) = 0$.

Conversely, suppose $\mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \in A) > 0) = 0$. Then for $B \in \mathcal{B}(\mathbb{X})$ with $0 < \lambda(B) < \infty$,

$$\begin{aligned} \mathbf{P}^{\mathbf{m}}(A) &= \frac{1}{\gamma \lambda(B)} \mathbf{E} \int_{\mathbb{X}} 1_{x \in B} 1_{\theta_x^{-1} \in A} \mathbf{m}(dx) \\ &= \frac{1}{\gamma \lambda(B)} \mathbf{E}[\mathbf{m}(x \in \mathbb{X} : x \in B, \theta_x^{-1} \in A)] \\ &\leq \frac{1}{\gamma \lambda(B)} \mathbf{E}[\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \in A)] \\ &= 0, \end{aligned}$$

completing the proof. □

Theorem 4.2.5. *Let $A \in \mathcal{A}$. Then the following are equivalent:*

- (a) $\mathbf{P}^{\mathbf{m}}(A) = 1$,
- (b) $\mathbf{P}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0) = 1$,
- (c) $\mathbf{P}^{\mathbf{m}}(\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0) = 1$.

Proof.

(a) \iff (b): This is the content of Lemma 4.2.4.

(b) \iff (c): This follows from Proposition 4.2.2 and the fact that the event $\{\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0\}$ contains $\{\mathbf{m} = \mathbf{0}\}$ and is shift-invariant: for all $y \in \mathbb{X}$,

$$\begin{aligned}
& \theta_y^{-1}\omega \in \{\mathbf{m}(x \in \mathbb{X} : \theta_x^{-1} \notin A) = 0\} \\
& \iff \mathbf{m}(\theta_y^{-1}\omega, \{x \in \mathbb{X} : \theta_x^{-1}\theta_y^{-1}\omega \notin A\}) = 0 \\
& \iff \mathbf{m}(\omega, \{yx : x \in \mathbb{X}, \theta_{yx}^{-1}\omega \notin A\}) \\
& \iff \mathbf{m}(\omega, \{x \in \mathbb{X}, \theta_x^{-1}\omega \notin A\}) = 0 \\
& \iff \omega \in \{\mathbf{m}(x \in \mathbb{X}, \theta_x^{-1}\omega \notin A) = 0\}.
\end{aligned}$$

□

Example 4.2.6. Fix some measurable space (S, Σ) and a measurable $f : \Omega \rightarrow S$. Define $F : \Omega \times \mathbb{X} \rightarrow S$ by $F(\omega, x) := f(\theta_x^{-1}\omega)$ for all $\omega \in \Omega, X \in \mathbf{m}(\omega)$, and $F(\omega, x)$ may be defined arbitrarily otherwise. It will be shown that knowing F up to a \mathbf{P} - or $\mathbf{P}^{\mathbf{m}}$ -null set on the support of \mathbf{m} is equivalent to knowing f up to a $\mathbf{P}^{\mathbf{m}}$ -null set. Indeed, suppose $f = f'$, $\mathbf{P}^{\mathbf{m}}$ -a.s., then it will be shown that the corresponding F, F' agree $\mathbf{P}, \mathbf{P}^{\mathbf{m}}$ -a.s. on the support of \mathbf{m} . By Theorem 1.2.3, \mathbf{P} - and $\mathbf{P}^{\mathbf{m}}$ -a.e. $\omega \in \Omega$ has for all $X \in \mathbf{m}(\omega)$ that $f(\theta_X^{-1}\omega) = f'(\theta_X^{-1}\omega)$, i.e. $F(\omega, X) = F'(\omega, X)$. Similarly, if either \mathbf{P} -a.e. or $\mathbf{P}^{\mathbf{m}}$ -a.e. $\omega \in \Omega$ is such that $F(\omega, X) = F'(\omega, X)$ for all $X \in \mathbf{m}(\omega)$, then

$$\begin{aligned}
f(\theta_X^{-1}\omega) &= F(\omega, X) \\
&= F'(\omega, X) \\
&= f'(\theta_X^{-1}\omega),
\end{aligned}$$

for \mathbf{P} -a.e. or $\mathbf{P}^{\mathbf{m}}$ -a.e. $\omega \in \Omega, X \in \mathbf{m}(\omega)$, so by Theorem 1.2.3 one finds that $f = f'$, $\mathbf{P}^{\mathbf{m}}$ -a.s. Thus, f may be defined under $\mathbf{P}^{\mathbf{m}}$ or F may be defined under \mathbf{P} or $\mathbf{P}^{\mathbf{m}}$, whichever is more convenient.

Theorem 4.2.7 (Slivnyak-Mecke Theorem). [4] *Let \mathbf{m} be a stationary random measure with intensity $\gamma \in (0, \infty)$ on \mathbb{X} . Then the distribution of \mathbf{m} under $\mathbf{P}^{\mathbf{m}}$ is the same as the distribution of $\mathbf{m} + \delta_e$ under \mathbf{P} iff \mathbf{m} is a homogeneous Poisson point process with intensity γ under \mathbf{P} .*

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